arXiv:1508.00187v3 [math.CO] 15 Nov 2016

THE NUMBERS OF EDGES OF THE ORDER POLYTOPE AND THE CHAIN POLYTOPE OF A FINITE PARTIALLY ORDERED SET

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ABSTRACT. Let P be an arbitrary finite partially ordered set. It will be proved that the number of edges of the order polytope $\mathscr{O}(P)$ is equal to that of the chain polytope $\mathscr{C}(P)$. Furthermore, it will be shown that the degree sequence of the finite simple graph which is the 1-skeleton of $\mathscr{O}(P)$ is equal to that of $\mathscr{C}(P)$ if and only if $\mathscr{O}(P)$ and $\mathscr{C}(P)$ are unimodularly equivalent.

INTRODUCTION

In [5] the combinatorial structure of the order polytope $\mathscr{O}(P)$ and the chain polytope $\mathscr{C}(P)$ of a finite poset (partially ordered set) P is studied in detail. Furthermore, the problem when $\mathscr{O}(P)$ and $\mathscr{C}(P)$ are unimodularly equivalent is solved in [3]. In this paper it is proved that, for an arbitrary finite poset P, the number of edges of the order polytope $\mathscr{O}(P)$ is equal to that of the chain polytope $\mathscr{C}(P)$. Furthermore, it is shown that the degree sequence of the finite simple graph which is the 1-skeleton of $\mathscr{O}(P)$ is equal to that of $\mathscr{C}(P)$ if and only if $\mathscr{O}(P)$ and $\mathscr{C}(P)$ are unimodularly equivalent.

1. EDGES OF ORDER POLYTOPES AND CHAIN POLYTOPES

Let $P = \{x_1, \ldots, x_d\}$ be a finite poset. Given a subset $W \subset P$, we introduce $\rho(W) \in \mathbb{R}^d$ by setting $\rho(W) = \sum_{i \in W} \mathbf{e}_i$, where $\mathbf{e}_1, \mathbf{e}_2 \ldots, \mathbf{e}_d$ are the canonical unit coordinate vectors of $\mathbb{R}^d$. In particular $\rho(\emptyset)$ is the origin of $\mathbb{R}^d$. A *poset ideal* of P is a subset I of P such that, for all x_i and x_j with $x_i \in I$ and $x_j \leq x_i$, one has $x_j \in I$. An *antichain* of P is a subset A of P such that x_i and x_j belonging to A with $i \neq j$ are incomparable. The empty set $\emptyset$ is a poset ideal as well as an antichain of P. We say that x_j *covers* x_i if $x_i < x_j$ and $x_i < x_k < x_j$ for no $x_k \in P$. A chain $x_{j_1} < x_{j_2} < \cdots < x_{j_\ell}$ of P is called *saturated* if x_{j_q} covers $x_{j_{q-1}}$ for $1 < q \leq \ell$.

The *order polytope* of P is the convex polytope $\mathscr{O}(P) \subset \mathbb{R}^d$ which consists of those $(a_1, \ldots, a_d) \in \mathbb{R}^d$ such that $0 \leq a_i \leq 1$ for every $1 \leq i \leq d$ together with

$$a_i \geq a_j$$

if $x_i \leq x_j$ in P.

The *chain polytope* of P is the convex polytope $\mathscr{C}(P) \subset \mathbb{R}^d$ which consists of those $(a_1, \ldots, a_d) \in \mathbb{R}^d$ such that $a_i \geq 0$ for every $1 \leq i \leq d$ together with

$$a_{i_1} + a_{i_2} + \cdots + a_{i_k} \leq 1$$

for every maximal chain $x_{i_1} < x_{i_2} < \cdots < x_{i_k}$ of P.

2010 Mathematics Subject Classification: Primary 52B05; Secondary 06A07.
Key words and phrases: chain polytope, order polytope, partially ordered set.

One has $\dim \mathscr{O}(P) = \dim \mathscr{C}(P) = d$. The vertices of $\mathscr{O}(P)$ is those $\rho(I)$ for which I is a poset ideal of P ([5, Corollary 1.3]) and the vertices of $\mathscr{C}(P)$ is those $\rho(A)$ for which A is an antichain of P ([5, Theorem 2.2]). It then follows that the number of vertices of $\mathscr{O}(P)$ is equal to that of $\mathscr{C}(P)$. Furthermore, the volume of $\mathscr{O}(P)$ and that of $\mathscr{C}(P)$ are equal to $e(P)/d!$, where $e(P)$ is the number of linear extensions of P ([5, Corollary 4.2]).

In [4] a characterization of edges of $\mathscr{O}(P)$ and those of $\mathscr{C}(P)$ is obtained. Recall that a subposet Q of a finite poset P is said to be *connected* in P if, for each x and y belonging to Q, there exists a sequence $x = x_0, x_1, \ldots, x_s = y$ with each $x_i \in Q$ for which x_{i-1} and x_i are comparable in P for each $1 \leq i \leq s$.

Lemma 1.1. *Let P be a finite poset.*

(a) *Given poset ideals I and J with $I \neq J$, the segment combining $\rho(I)$ with $\rho(J)$ is an edge of $\mathscr{O}(P)$ if and only if $I \subset J$ and $J \setminus I$ is connected in P.*

(b) *Given antichains A and B with $A \neq B$, the segment combining $\rho(A)$ with $\rho(B)$ is an edge of $\mathscr{C}(P)$ if and only if $(A \setminus B) \cup (B \setminus A)$ is connected in P.*

Let, in general, G be a finite simple graph, i.e., a finite graph with no loop and with no multiple edge, on the vertex set $V(G) = \{v_1, \ldots, v_n\}$. The *degree* $\deg_G(v_i)$ of each $v_i \in V(G)$ is the number of edges e of G with $v_i \in e$. Let $i_1 \cdots i_n$ denote a permutation of $1, \ldots, n$ for which $\deg_G(v_{i_1}) \leq \cdots \leq \deg_G(v_{i_n})$. The *degree sequence* ([1, p. 216]) of G is the finite sequence $(\deg_G(v_{i_1}), \ldots, \deg_G(v_{i_n}))$.

Example 1.2. Let X denote the poset

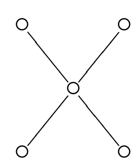

Figure 1

Then the degree sequence of the finite simple graph which is the 1-skeleton of $\mathscr{O}(X)$ is

$$(6,6,6,6,6,6,6,6)$$

and that of $\mathscr{C}(X)$ is

$$(5,6,6,6,6,6,6,7).$$

This observation guarantees that, even though the number of edges of $\mathscr{O}(X)$ is equal to that of $\mathscr{C}(X)$, one cannot construct a bijection $\varphi : V(\mathscr{O}(X)) \to V(\mathscr{C}(X))$, where $V(\mathscr{O}(X))$ is the set of vertices of $\mathscr{O}(X)$ and $V(\mathscr{C}(X))$ is that of $\mathscr{C}(X)$, with the property that, for α and β belonging to $V(\mathscr{O}(X))$, the segment combining α and β is an edge of $\mathscr{O}(X)$ if and only if the segment combining $\varphi(\alpha)$ and $\varphi(\beta)$ is an edge of $\mathscr{C}(X)$.

2. THE NUMBER OF EDGES OF ORDER POLYTOPES AND CHAIN POLYTOPES

We now come to the main result of the present paper.

Theorem 2.1. *Let P be an arbitrary finite poset. Then the number of edges of the order polytope $\mathscr{O}(P)$ is equal to that of the chain polytope $\mathscr{C}(P)$.*

Proof. Let Ω denote the set of pairs (I,J), where I and J are poset ideals of P with $I \neq J$ for which $I \subset J$ and $J \setminus I$ is connected in P. Let Ψ denote the set of pairs (A,B), where A and B are antichains of P with $A \neq B$ for which $(A \setminus B) \cup (B \setminus A)$ is connected in P.

As is stated in the proof of [4, Lemma 2.3], if there exist $x, x' \in A$ and $y, y' \in B$ with $x < y$ and $y' < x'$, then $(A \setminus B) \cup (B \setminus A)$ cannot be connected. In fact, if $(A \setminus B) \cup (B \setminus A)$ is connected, then there exists a sequence $x = x_0, y_0, x_1, y_1, \ldots, y_s, x_s = x'$ with each $x_i \in A \setminus B$ and each $b_j \in B \setminus A$ such that x_i and y_i are comparable for each i and that y_j and x_{j+1} are comparable for each j. Since $x < y$ and since B is an antichain, it follows that $x = x_0 < y_0$. Then, since A is an antichain, one has $y_0 > x_1$. Continuing these arguments says that $y_s > x_s = x'$. However, since $y' < x'$, one has $y' < y_s$, which contradicts the fact that B is an antichain.

As a result, each $(A,B) \in \Psi$ can be required to satisfy either (i) $B \subset A$ or (ii) $b < a$ whenever $a \in A$ and $b \in B$ are comparable. By virtue of Lemma 1.1, our work is to construct a bijection between Ω and Ψ.

Given $(I,J) \in \Omega$, we associate with

$$A = \max(J), \;\; B = \min(J \setminus I) \cup (\max(I) \cap \max(J))$$

with setting $\min(J \setminus I) = \emptyset$ if $|J \setminus I| = 1$, where, say, $\max(I)$ (resp. $\min(I)$) stands for the set of maximal (resp. minimal) elements of I. It then follows that

$$\min(J \setminus I) \cap (\max(I) \cap \max(J)) = \emptyset. \tag{1}$$

Now, $A = \max(J)$ is an antichain of P. If $x \in \min(J \setminus I)$ and $y \in \max(I) \cap \max(J)$, then $x \not\leq y$ since $x \notin I$ and $y \in I$, and $y \not\leq x$ since $x \in J$, $x \neq y$ and $y \in \max(J)$. Hence B is an antichain of P. Furthermore, since $\max(J) \cap \min(J \setminus I) = \emptyset$, where $\min(J \setminus I) = \emptyset$ if $|J \setminus I| = 1$, it follows that $A \setminus B = \max(J) \setminus \max(I) = \max(J \setminus I)$ and $B \setminus A = \min(J \setminus I)$. Hence $(A \setminus B) \cup (B \setminus A)$ is connected in P. Thus $(A,B) \in \Psi$.

We claim that the above map which associates $(I,J) \in \Omega$ with $(A,B) \in \Psi$ is, in fact, a bijection between Ω and Ψ.

Let (I,J) and (I',J') belong to Ω with $\max(J) = \max(J')$ and

$$\min(J \setminus I) \cup (\max(I) \cap \max(J)) = \min(J' \setminus I') \cup (\max(I') \cap \max(J')). \tag{2}$$

Then $J = J'$. Let $\max(I) \cap \max(J) \neq \max(I') \cap \max(J)$ and, say, $\max(I) \cap \max(J) \neq \emptyset$. Let $x \in \max(I) \cap \max(J)$ and $x \notin \max(I') \cap \max(J)$. By using (2), one has $x \in \min(J \setminus I')$. Since $\max(J \setminus I') \cap \min(J \setminus I') = \emptyset$, where $\min(J \setminus I) = \emptyset$ if $|J \setminus I| = 1$, there is $y \in \max(J \setminus I')$ with $x < y$. This is impossible since x and y belong to $\max(J)$. As a result, one has $\max(I) \cap \max(J) = \max(I') \cap \max(J)$. It then follows from (1) and (2) that $\min(J \setminus I) = \min(J \setminus I')$. In addition,

$$\max(J \setminus I) = \max(J) \setminus \max(I) = \max(J) \setminus (\max(I) \cap \max(J)) = \max(J \setminus I').$$

Since

$$J \setminus I = \{\, x \in P \,:\, x \leq b,\, \exists b \in \max(J \setminus I) \,\} \bigcap \{\, x \in P \,:\, a \leq x,\, \exists a \in \min(J \setminus I) \,\},$$

it follows from $\min(J \setminus I) = \min(J \setminus I')$ and $\max(J \setminus I) = \max(J \setminus I')$ that $J \setminus I = J \setminus I'$. Hence $I = I'$ and $(I,J) = (I',J')$, as desired.

Let (A,B) belong to Ψ. Let J be the poset ideal of P with $\max(J) = A$. Let I be the poset ideal of P consisting of those $x \in J$ for which $x \geq y$ for no $y \in B \setminus A$. In particular, $I = J \setminus \{x\}$ if $B \subset A$ with $A \setminus B = \{x\}$. Then $\max(J \setminus I) = A \setminus B$ and $\min(J \setminus I) = B \setminus A$, where $\min(J \setminus I) = \emptyset$ if $|J \setminus I| = 1$. Hence $I \subset J$ and $J \setminus I$ is connected in P. Furthermore, $B = \min(J \setminus I) \cup (\max(I) \cap \max(J))$, as required. □

3. Degree sequences of 1-skeletons of order and chain polytopes

Let $\mathbb{Z}^{d \times d}$ denote the set of $d \times d$ integral matrices. A matrix $A \in \mathbb{Z}^{d \times d}$ is *unimodular* if $\det(A) = \pm 1$. Given integral polytopes $\mathscr{P} \subset \mathbb{R}^d$ of dimension d and $\mathscr{Q} \subset \mathbb{R}^d$ of dimension d, we say that $\mathscr{P}$ and $\mathscr{Q}$ are *unimodularly equivalent* if there exists a unimodular matrix $U \in \mathbb{Z}^{d \times d}$ and an integral vector $\mathbf{w} \in \mathbb{Z}^d$ such that $Q = f_U(P) + \mathbf{w}$, where f_U is the linear transformation of $\mathbb{R}^d$ defined by U, i.e., $f_U(\mathbf{v}) = \mathbf{v}U$ for all $\mathbf{v} \in \mathbb{R}^d$.

Recall from [3] that $\mathscr{O}(P)$ and $\mathscr{C}(P)$ are unimodularly equivalent if and only if the poset X of Figure 1 does not appear as a subposet of P. In consideration of Example 1.2, we now prove the following

Theorem 3.1. *Let P be a finite poset. Then the degree sequence of the finite simple graph which is the 1-skeleton of $\mathscr{O}(P)$ is equal to that of $\mathscr{C}(P)$ if and only if $\mathscr{O}(P)$ and $\mathscr{C}(P)$ are unimodularly equivalent.*

Proof. **("If")** If $\mathscr{O}(P)$ and $\mathscr{C}(P)$ are unimodularly equivalent, then the 1-skeleton of $\mathscr{O}(P)$ is isomorphic to that of $\mathscr{C}(P)$ as finite graphs. Thus in particular the degree sequence of the 1-skeleton of $\mathscr{O}(P)$ is equal to that of $\mathscr{C}(P)$, as required.

("Only If") Let $|P| = d$. Suppose that $\mathscr{O}(P)$ is not unimodularly equivalent to $\mathscr{C}(P)$. It then follows from [3, Theorem 2.1] that the poset X of Figure 1 does appear as a subposet of P. Let $X = \{a,b,c,g,h\}$, where $a < c, b < c, c < g$ and $c < h$. Work with the same notation as in the proof of Theorem 2.1. Write $G_{\mathscr{O}(P)}$ for the finite simple graph which is the 1-skeleton of $\mathscr{O}(P)$ and $G_{\mathscr{C}(P)}$ for that of $\mathscr{C}(P)$.

Let $A \neq \emptyset$ be an antichain of P. Then $(\emptyset, A) \in \Psi$ if and only if $|A| = 1$. It then follows that the degree of the vertex $\rho(\emptyset)$ of $G_{\mathscr{C}(P)}$ is equal to d.

We now prove that the degree of each vertex of $G_{\mathscr{O}(P)}$ is at least $d+1$. Let I be a poset ideal of P. For each $x \in I$ we write I' for the poset ideal of P consisting of those $y \in I$ with $y \not\geq x$. Then $(I', I) \in \Omega$. For each $x \in P \setminus I$ we write I' for the poset ideal of P consisting of those $y \in P$ with either $y \in I$ or $y \leq x$. Then $(I, I') \in \Omega$. As a result, the degree of each vertex of $G_{\mathscr{O}(P)}$ is at least d.

Since the poset $X = \{a,b,c,g,h\}$ of Figure 1 does appear as a subposet of P, one has either $c \in I$ or $c \notin I$. Let $c \in I$ and I' the poset ideal of P consisting of those $y \in I$ with neither $y \geq a$ nor $y \geq b$. Then $(I', I) \in \Omega$. Let $c \notin I$ and I' the poset ideal of P consisting of those $y \in P$ with $y \in I$ or $y \leq g$ or $y \leq h$. Then $(I, I') \in \Omega$. Hence the degree of each vertex of $G_{\mathscr{O}(P)}$ is at least $d+1$, as desired. □

Together with [3, Corollary 2.3] it follows that

Corollary 3.2. *Given a finite poset P, the following conditions are equivalent:*

(i) $\mathscr{O}(P)$ *and* $\mathscr{C}(P)$ *are unimodularly equivalent;*
(ii) $\mathscr{O}(P)$ *and* $\mathscr{C}(P)$ *are affinely equivalent;*
(iii) $\mathscr{O}(P)$ *and* $\mathscr{C}(P)$ *have the same* f*-vector* ([2, p. 12])*;*
(iv) The number of facets of $\mathscr{O}(P)$ *is equal to that of* $\mathscr{C}(P)$*;*
(v) the degree sequence of the finite simple graph which is the 1*-skeleton of* $\mathscr{O}(P)$ *is equal to that of* $\mathscr{C}(P)$*;*
(vi) The poset X *of Figure* 1 *of does not appear as a subposet of* P.

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